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4.4.B17f
Ve vektorovém prostoru V jsou zadány podprostory W_1, W_2.
Nalezněte dimenzi a bázi podprostorů $W_1,\ W_2,\ W_1 + W_2,\ W_1 \cap W_2$, je-li:

$$V = \mathbb{R}^5;\ W_1 = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4],\ W_2 = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4],\ \text{kde}$$

$$\begin{array}{llll} \mathbf{u}_1 = (1,-2,0,3,5), & \mathbf{u}_2 = (-1,3,2,-5,-9), & \mathbf{u}_3 = (2,-3,-2,0,3), & \mathbf{u}_4 = (-1,1,2,3,2) \\ \mathbf{v}_1 = (1,1,1,1,1), & \mathbf{v}_2 = (-2,2,0,2,1), & \mathbf{v}_3 = (0,4,2,4,3), & \mathbf{v}_4 = (-1,7,3,7,5) \end{array}$$

1 Báze a dimenze W_1

$$\begin{matrix}\mathbf{u}_1\\\mathbf{u}_2\\\mathbf{u}_3\\\mathbf{u}_4\end{matrix}\begin{pmatrix} 1 & -2 & 0 & 3 & 5 \\ -1 & 3 & 2 & -5 & -9 \\ 2 & -3 & -2 & 0 & 3 \\ -1 & 1 & 2 & 3 & 2 \end{pmatrix} \sim \begin{matrix}(1)\\(1)+(2)\\-2\cdot(1)+(3)\\(1)+(4)\end{matrix}\begin{pmatrix} 1 & -2 & 0 & 3 & 5 \\ 0 & 1 & 2 & -2 & -4 \\ 0 & 1 & -2 & -6 & -7 \\ 0 & -1 & 2 & 6 & 7 \end{pmatrix} \overset{(3)=-(4)}{\sim}$$

$$\begin{matrix}(1)\\(2)\\(2)+(4)\end{matrix}\begin{pmatrix} 1 & -2 & 0 & 3 & 5 \\ 0 & 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 & 3 \end{pmatrix}$$

$$\dim(W_1) = 3$$

Báze W_1 např.:

$$\mathbf{a}_1 = (1,-2,0,3,5), \quad \mathbf{a}_2 = (0,1,2,-2,-4), \quad \mathbf{a}_3 = (0,0,4,4,3)$$

2 Báze a dimenze W_2

$$\begin{matrix}\mathbf{v}_1\\\mathbf{v}_2\\\mathbf{v}_3\\\mathbf{v}_4\end{matrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & 2 & 1 \\ 0 & 4 & 2 & 4 & 3 \\ -1 & 7 & 3 & 7 & 5 \end{pmatrix} \sim \begin{matrix}(1)\\2\cdot(1)+(2)\\(3)\\(1)+(4)\end{matrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & 4 & 3 \\ 0 & 4 & 2 & 4 & 3 \\ 0 & 8 & 4 & 8 & 6 \end{pmatrix} \overset{(2)=(3)=\frac{1}{2}(4)}{\sim} \begin{matrix}(1)\\(2)\end{matrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & 4 & 3 \end{pmatrix}$$

$$\dim(W_2) = 2$$

Báze W_2 např.:

$$\mathbf{b}_1 = (1,1,1,1,1), \quad \mathbf{b}_2 = (0,4,2,4,3)$$

3 Báze a dimenze $W_1 + W_2$

$$\begin{matrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\\\mathbf{b}_1\\\mathbf{b}_2\end{matrix}\begin{pmatrix}1&-2&0&3&5\\0&1&2&-2&-4\\0&0&4&4&3\\1&1&1&1&1\\0&4&2&4&3\end{pmatrix}\sim\begin{matrix}(4)\\(2)\\-1\cdot(4)+(1)\\(5)\\(3)\end{matrix}\begin{pmatrix}1&1&1&1&1\\0&1&2&-2&-4\\0&-3&-1&2&4\\0&4&2&4&3\\0&0&4&4&3\end{pmatrix}\sim$$

$$\begin{matrix}(1)\\(2)\\3\cdot(2)+(3)\\-4\cdot(2)+(4)\\(5)\end{matrix}\begin{pmatrix}1&1&1&1&1\\0&1&2&-2&-4\\0&0&5&-4&-8\\0&0&-6&12&19\\0&0&4&4&3\end{pmatrix}\sim\begin{matrix}(1)\\(2)\\(3)\\\frac{6}{5}\cdot(3)+(4)\\-\frac{4}{5}\cdot(3)+(5)\end{matrix}\begin{pmatrix}1&1&1&1&1\\0&1&2&-2&-4\\0&0&5&-4&-8\\0&0&0&\frac{36}{5}&\frac{47}{5}\\0&0&0&\frac{36}{5}&\frac{47}{5}\end{pmatrix}\overset{(4)=(5)}{\sim}$$

$$\begin{matrix}(1)\\(2)\\(3)\\5\cdot(4)\end{matrix}\begin{pmatrix}1&1&1&1&1\\0&1&2&-2&-4\\0&0&5&-4&-8\\0&0&0&36&47\end{pmatrix}$$

$\dim(W_1 + W_2) = 4$

Báze $W_1 + W_2$ např.:

$\mathbf{c}_1 = (1,1,1,1,1), \quad \mathbf{c}_2 = (0,1,2,-2,-4), \quad \mathbf{c}_3 = (0,0,5,-4,-8), \quad \mathbf{c}_4 = (0,0,0,36,47)$

4 Báze a dimenze $W_1 \cap W_2$

$\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) = 3 + 2 - 4 = 1$

$\mathbf{x} \in W_1 \cap W_2 \Rightarrow \mathbf{x} \in W_1 \ \wedge \ \mathbf{x} \in W_2 \Rightarrow$

$\mathbf{x} = t_1 \cdot \mathbf{a}_1 + t_2 \cdot \mathbf{a}_2 + t_3 \cdot \mathbf{a}_3 \ \wedge \ \mathbf{x} = s_1 \cdot \mathbf{b}_1 + s_2 \cdot \mathbf{b}_2,$ kde $t_1, t_2, t_3, s_1, s_2 \in \mathbb{R} \Rightarrow$

$$t_1 \cdot \mathbf{a}_1 + t_2 \cdot \mathbf{a}_2 + t_3 \cdot \mathbf{a}_3 - s_1 \cdot \mathbf{b}_1 - s_2 \cdot \mathbf{b}_2 = \mathbf{o}$$

$$t_1\cdot(1,-2,0,3,5)+t_2\cdot(0,1,2,-2,-4)+t_3\cdot(0,0,4,4,3)-s_1\cdot(1,1,1,1,1)-s_2\cdot(0,4,2,4,3) = (0,0,0,0,0)$$

$$(t_1-s_1, -2t_1+t_2-s_1-4s_2, 2t_2+4t_3-s_1-2s_2, 3t_1-2t_2+4t_3-s_1-4s_2, 5t_1-4t_2+3t_3-s_1-3s_2) = (0,0,0,0,0)$$

$$\begin{array}{rcrcrcrcrcl} t_1 & & & & & - & s_1 & & & = & 0\\ -2t_1 & + & t_2 & & & - & s_1 & - & 4s_2 & = & 0\\ & & 2t_2 & + & 4t_3 & - & s_1 & - & 2s_2 & = & 0\\ 3t_1 & - & 2t_2 & + & 4t_3 & - & s_1 & - & 4s_2 & = & 0\\ 5t_1 & - & 4t_2 & + & 3t_3 & - & s_1 & - & 3s_2 & = & 0 \end{array}$$

$$\left(\begin{array}{rrrrr|r}1&0&0&-1&0&0\\-2&1&0&-1&-4&0\\0&2&4&-1&-2&0\\3&-2&4&-1&-4&0\\5&-4&3&-1&-3&0\end{array}\right)\sim\begin{matrix}(1)\\2\cdot(1)+(2)\\(3)\\-3\cdot(1)+(4)\\-5\cdot(1)+(5)\end{matrix}\left(\begin{array}{rrrrr|r}1&0&0&-1&0&0\\0&1&0&-3&-4&0\\0&2&4&-1&-2&0\\0&-2&4&2&-4&0\\0&-4&3&4&-3&0\end{array}\right)\sim$$

$$\begin{matrix}(1)\\(2)\\-2\cdot(2)+(3)\\2\cdot(2)+(4)\\4\cdot(2)+(5)\end{matrix}\left(\begin{array}{rrrrr|r}1&0&0&-1&0&0\\0&1&0&-3&-4&0\\0&0&4&5&6&0\\0&0&4&-4&-12&0\\0&0&3&-8&-19&0\end{array}\right)\sim\begin{matrix}(1)\\(2)\\(3)\\-(3)+(4)\\-\frac{3}{4}\cdot(3)+(5)\end{matrix}\left(\begin{array}{rrrrr|r}1&0&0&-1&0&0\\0&1&0&-3&-4&0\\0&0&4&5&6&0\\0&0&0&-9&-18&0\\0&0&0&-\frac{47}{4}&-\frac{94}{4}&0\end{array}\right)\overset{(5)=\frac{36}{47}(4)}{\sim}$$

$$\begin{matrix}(1)\\(2)\\(3)\\-\frac{1}{9}\cdot(4)\end{matrix}\left(\begin{array}{rrrrr|r}1&0&0&-1&0&0\\0&1&0&-3&-4&0\\0&0&4&5&6&0\\0&0&0&1&2&0\end{array}\right)\Rightarrow\begin{array}{l}s_2 = k\\ s_1 = -2s_2 = -2k\end{array}$$

$\mathbf{x} = -2k \cdot (1,1,1,1,1) + k \cdot (0,4,2,4,3) = k \cdot (-2,2,0,2,1)$

Báze $W_1 \cap W_2$ např.:

$\mathbf{d}_1 = (-2,2,0,2,1)$